Boston Area Undergraduate
Physics Competition

Solutions

1. Let the friction force on the ball be $F$. Then $F$ must cancel the component of gravity in the tangential direction; thus $F = mg\sin\theta$.

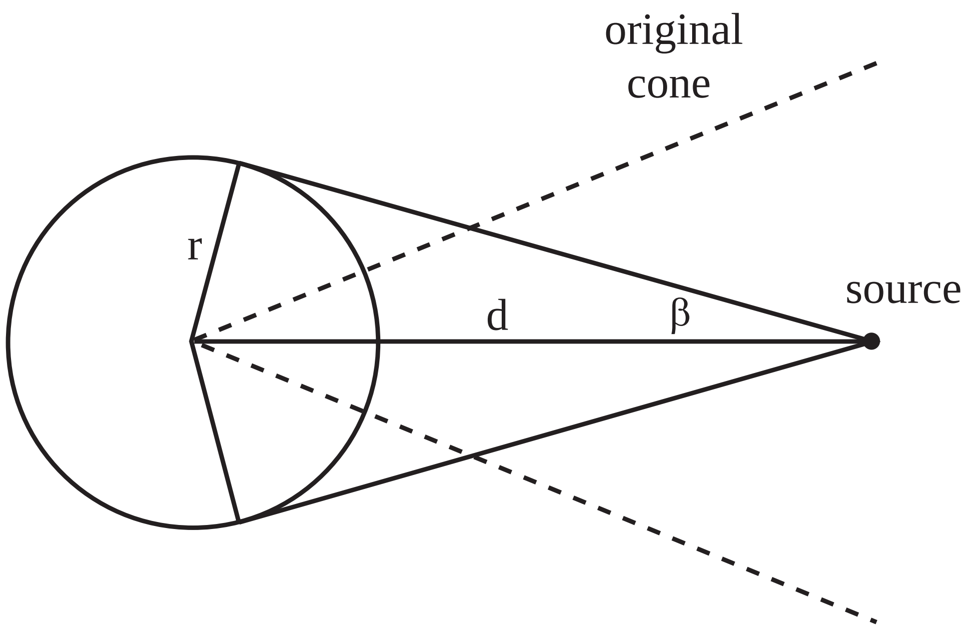
   The torque on the ball is $\tau = Fr$. Using $F = mg\sin\theta$, we get $\tau = mgr\sin\theta$. This torque must equal $I\alpha_b$, where $\alpha_b$ is the angular acceleration of the ball, which is related to the $\alpha$ of the cylinder by $\alpha_b = (R/r)\alpha$. Thus, $\tau = I\alpha_b$ gives

$$mgr\sin\theta = \left(\frac{2}{5}mr^2\right)\left(\frac{R}{r}\alpha\right) \qquad \Longrightarrow \qquad \alpha = \frac{5g\sin\theta}{2R}. \tag{1}$$

2. When the light bounces off the surface, its reflected angle is equal to its incident angle. Therefore, we may use the method of images to determine if a particular ray of light hits the receiver.

   If we look at a two-dimensional cross-section of the cone, we see that when the picture is repeatedly reflected across the edge of the cone, the receiver turns into a full circle of radius $r$. The method of images therefore tells us that in the original three-dimensional case, the given receiver turns into a full sphere of radius $r$, centered at the vertex of the cone.

   The given problem therefore reduces to the problem: What fraction of light emitted from a source falls on a sphere of radius $r$ centered at a point a distance $d$ from the source? (Note that the given vertex angle $\theta$ is irrelevant.)




From the figure, we see that we need to find the fractional solid angle subtended by a cone with half-angle $\beta$, where $\sin\beta = r/d$. Looking at a sphere of radius $R$, the area of the spherical "cap" subtended by this cone can be found by slicing the cap into circular bands. If $\alpha$ describes the angle away from the top of the cap, then the corresponding circle has radius $2\pi(R\sin\alpha)$, so the resulting integral for the area of the cap is

$$A = \int_0^\beta (2\pi R\sin\alpha)(R\,d\alpha) = -2\pi R^2\cos\alpha\Big|_0^\beta = 2\pi R^2(1-\cos\beta). \quad (2)$$

The fraction of the total area is therefore

$$\text{Fraction} = \frac{A}{4\pi R^2} = \frac{1}{2}(1-\cos\beta) = \frac{1}{2}\left(1 - \frac{\sqrt{d^2-r^2}}{d}\right). \quad (3)$$

(If $r > d$, then the fraction equals 1, of course.)

3. The normal force from the plane is $N = mg\cos\theta$, so the friction force is $\mu N = mg\sin\theta$. This force acts in the direction opposite to the motion. There is also the gravitational force of $mg\sin\theta$ pointing down the plane.

The magnitudes of these two forces are equal, so the acceleration along the direction of motion equals the negative of the acceleration in the direction down the plane. Therefore, in a small increment of time, the speed that the block loses along its direction of motion exactly equals the speed that it gains in the direction down the plane. Letting $v$ be the speed of the block, and letting $v_y$ be the component of the speed in the direction down the plane, we therefore have

$$v + v_y = C, \tag{4}$$

where $C$ is a constant. $C$ is given by its initial value, which is $V+0 = V$ (where $V$ is the initial speed of the block). The final value of $C$ is $V_f + V_f = 2V_f$ (where $V_f$ is the final speed of the block), since the block is essentially moving straight down the plane after a very long time. Therefore,

$$2V_f = V \qquad \Longrightarrow \qquad V_f = V/2. \tag{5}$$

4. Start with the first law of thermodynamics (energy conservation),

$$dQ = dU - dW \tag{6}$$

where $dQ$ is an infinitesimal amount of heat added to the system, $dU$ is the change in internal energy, and $dW$ is an infinitesimal amount of mechanical work done on the system.

Consider going around any closed loop in the state of the system. By 'state' we mean pressure $p$, volume $V$ and temperature $T$. In our system, knowing any 2 of these determines the third; for instance $T$ is a function of $(p, V)$ which is given by the ideal gas law. $U$ is a function of the state alone, so adding up the $dU$ changes around the closed loop must give zero. Therefore for a single traversal of the loop, $\Delta Q = -\Delta W$. Mechanical work done is caused by changes in volume,

$$dW = -p\, dV, \tag{7}$$

so the integral of $dW$ around a closed loop is just the negative of the area enclosed on the $(p, V)$ plane (for a clockwise loop, as we have). For each traversal of the loop, therefore

$$Q_{\text{in}} - Q_{\text{out}} = \Delta Q = -\Delta W = \text{area enclosed in } (p, V) = \tfrac{1}{2} p_0 V_0. \tag{8}$$

We have given the constants $P$ and $V$ given in the problem the more convenient symbols $p_0$ and $V_0$.

We have found the numerator of the fraction giving the efficiency (if you used the methods below to find this numerator, that was fine too). Now all that remains is the denominator, $Q_{\rm in}$.

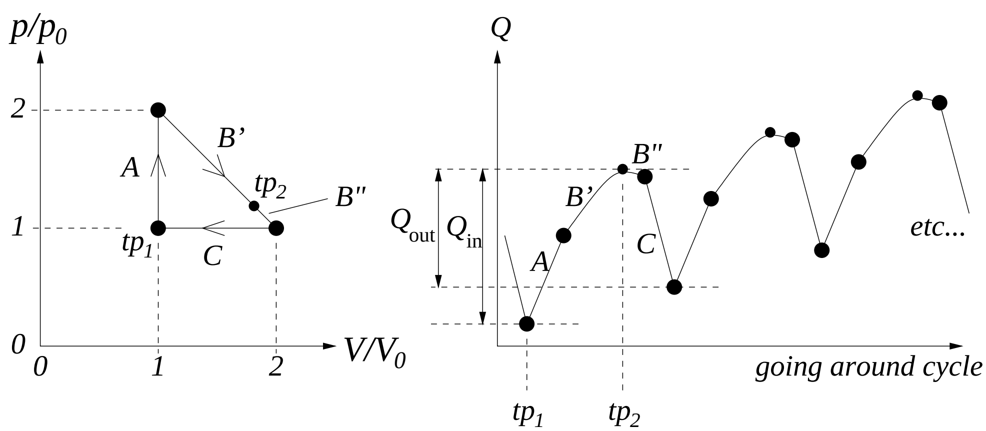
We're given the internal energy $U = \frac{3}{2}nkT$, from which you have to realise that $n$ is the *number of particles* not the number of moles of particles. This is obvious since $n$ is multiplied by $k$, Boltzmann's constant, rather than $R$, the gas constant. We have an ideal gas,

$$pV = NRT \text{ or equivalently, } pV = nkT, \tag{9}$$

where $N$ is the number of moles, and $n$ the number of particles (note this is swapped from the usual notation for $n$ and $N$). The second form is the useful one for us, since from it follows that internal energy can be written

$$U = \tfrac{3}{2}pV. \tag{10}$$

Now for any segment of the path we can find $\Delta U$, without explicit reference to $T$.




We need to consider the complete path, and find the 'turning points' where the flow of heat (sign of $dQ$) changes direction ($tp_1$ and $tp_2$ in the right-hand figure). $Q_{\rm in}$ is then the integral between these turning points along one route of the path (and $Q_{\rm out}$ is the same integral back along the other route). We need to find how $Q$ changes in each side of the triangle, labeled A,B and C (see left-hand figure):

**Leg A**: no volume change, so $\Delta W = 0$. Due to a doubling of pressure, $U$ has changed from $\frac{3}{2}p_0V_0$ to twice that, so $\Delta Q = \Delta U - \Delta W = +\frac{3}{2}p_0V_0$.

**Leg C**: no pressure change, so the work integral $\Delta W = -\int p\, dV$ is simply $-p\,\Delta V = p_0V_0$. $U$ goes from its value at the end of leg A back to its original value, so $\Delta U = -\frac{3}{2}p_0V_0$. Together these give $\Delta Q = -\frac{5}{2}p_0V_0$. Checking the signs here was important: the effects combine to give increased heat output.

From the above consideration of $dQ$ in legs A and C, it is clear that one turning point is $tp_1$ as shown.

**Leg B**: This is *not* an isothermal change (which would correspond to a hyperbola defined by $pV = \text{const}$, and would imply that $dQ > 0$ everywhere along the leg). It is also tempting to assume that since $\Delta U = 0$ along this leg (same start and end $T$), one can find $\Delta W$ and be done with the problem. Not so! Because $T$ (hence $U$) is dropping at the end of the leg while $W$ is also dropping (work is being done *by* the system), there is the possibility that $dQ$ changes sign along the leg, making $tp_2$ happen some fraction of the way along the leg.

We parametrize the leg by a unitless number $x = [1, 2]$, giving $p(x) = p_0(3 - x)$ and $V(x) = V_0x$. Therefore

$$dQ = dU - dW = \tfrac{3}{2}(p\,dV + V\,dP) + p\,dV = \frac{p_0V_0}{2}(15 - 8x)\,dx, \quad (11)$$

where the derivatives $dp = -p_0dx$ and $dV = V_0dx$ were used. Clearly dQ changes sign when $15 - 8x = 0$, that is, at $x = 15/8$. Note that this turning point is not simply at $x = 3/2$ (half-way through the leg), when $T$ reaches its maximum. The location of this turning point can also be found by considering the criterion for *adiabaticity* ($dQ = 0$), namely $dp/dV = -\gamma p/V = -\frac{5}{3}p/V$.

Splitting the leg into parts B′ and B″ at this turning point as shown, we need the heat input in leg B′ only,

$$\Delta Q = \int_{x=1}^{x=15/8} dQ = \frac{p_0V_0}{2}\int_{x=1}^{x=15/8}(15 - 8x)\,dx. \quad (12)$$

The $x$-integral gives $[15x - 4x^2]_1^{15/8} = \frac{49}{16}$ after a little simplification, so $\Delta Q = +\frac{49}{32}p_0V_0$.

Adding $\Delta Q$ from legs A and B gives $Q_{\rm in} = (\frac{3}{2} + \frac{49}{32})p_0 V_0 = \frac{97}{32}p_0 V_0$, an admittedly slightly messy fraction.

Finally the efficiency is

$$\epsilon = \frac{Q_{\rm in} - Q_{\rm out}}{Q_{\rm in}} = \frac{\frac{1}{2}p_0 V_0}{\frac{97}{32}p_0 V_0} = \frac{16}{97} = 0.1649\cdots \text{ or about } 16.5\% \tag{13}$$

Note that this is very close to the incorrect answer of 1/6 obtained if $tp_2$ is assumed to be at the lower right vertex.

5. At time $t$, the movable end of the band is a distance $\ell(t) = L + Vt$ from the wall. Let the ant's distance from the wall be $r(t)$.

   Consider the fraction of the ant's position along the band, $F(t) = r(t)/\ell(t)$. The given question is equivalent to: For what value of $t$ does the fraction, $F(t)$, become zero (if at all)? Let us see how $F(t)$ changes with time.

   After an infinitesimal time, $dt$, the ant's position, $r$, increases by $(r/\ell)V\,dt$ due to the stretching, and decreases by $u\,dt$ due to the crawling. Therefore,

$$\begin{aligned} F(t+dt) &= \frac{r + (r/\ell)V\,dt - u\,dt}{\ell + V\,dt} \\ &= \frac{r}{\ell} - \frac{u\,dt}{\ell + V\,dt}. \end{aligned} \tag{14}$$

   To first order in $dt$, this yields

$$F(t+dt) = F(t) - \frac{u}{\ell}dt. \tag{15}$$

   In other words, $F(t)$ decreases due to the fact that in a time $dt$ the ant crawls a distance $u\,dt$ relative to the band, which has a length approximately $\ell(t)$. Eq. (15) gives

$$\frac{dF(t)}{dt} = -\frac{u}{\ell}. \tag{16}$$

   Using $\ell(t) = L + Vt$ and integrating eq. (16), we obtain

$$F(t) = 1 - \frac{u}{V}\ln\left(1 + \frac{V}{L}t\right), \tag{17}$$

   where the constant of integration has been chosen to satisfy $F(0) = 1$.

We now note that for *any* positive value for $u$, we can make $F(t) = 0$ by choosing

$$t = \frac{L}{V}\left(e^{V/u} - 1\right). \tag{18}$$

For very large $V/u$, the time it takes the ant to reach the wall becomes exponentially large, but it does indeed reach it in a finite time.

For very small $V/u$, (18) reduces to $t \approx L/u$ (using $e^x \approx 1 + x$), as it should.

6. A few simple examples suggest that the answer to the problem is $(N-1)\,\Omega$. Let's prove this in general. (We will use a superposition argument.)

Consider two points, $A$ and $B$, that are connected by one of the $1\,\Omega$ resistors. If we put a current $I$ in at $A$, and take a current $I$ out at $B$, then the effective resistance between $A$ and $B$ is $V/I$, where $V$ is the potential difference between the two points.

The situation where a current $I$ goes in at $A$ and out at $B$ can be considered as the superposition of two setups: (1) Put a current $\frac{N-1}{N}I$ in at $A$ and take a current $\frac{1}{N}I$ out at each of the other $N-1$ points, and (2) Take a current $\frac{N-1}{N}I$ out at $B$ and put a current $\frac{1}{N}I$ in at each of the other $N-1$ points.

In the first setup, let the current going from $A$ to $B$ be $I^A_{A\to B}$. In the second setup, let the current going from $A$ to $B$ be $I^B_{A\to B}$. (The superscript here denotes the point at which the current of $\frac{N-1}{N}I$ enters or leaves.) Then in the combined setup, the current going from $A$ to $B$ is $I^A_{A\to B} + I^B_{A\to B}$. Since this current passes along a $1\,\Omega$ resistor, the voltage difference between $A$ and $B$ is $V = (I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)$. The effective resistance between $A$ and $B$ is therefore

$$R_{AB} = (I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)/I. \tag{19}$$

We must now add up these $R_{AB}$ contributions from all of the resistors. Let the desired sum be $S$. Then

$$S = \sum_{A,B}(I^A_{A\to B} + I^B_{A\to B})(1\,\Omega)/I, \tag{20}$$

where the sum runs over all pairs of points $A, B$ that are connected by a resistor. By reversing the roles of $A$ and $B$, we may also write

$$S = \sum_{A,B}(I^B_{B\to A} + I^A_{B\to A})(1\,\Omega)/I. \tag{21}$$

Adding the two previous equations gives

$$2S = \sum_{A,B} (I^A_{A\to B} + I^B_{B\to A})(1\,\Omega)/I + \sum_{A,B} (I^A_{B\to A} + I^B_{A\to B})(1\,\Omega)/I. \quad (22)$$

The first sum is simply the sum of the $N$ currents entering the $N$ points in all of the $N$ setups of type "1" above. Since a current of $\frac{N-1}{N}I$ enters each point (by construction), the first sum equals $(N-1)(1\,\Omega)$. Likewise, the second sum deals with the $N$ currents leaving the $N$ points in all of the $N$ setups of type "2" above, so it also equals $(N-1)(1\,\Omega)$. Eq. (22) therefore gives

$$S = (N-1)\,\Omega. \quad (23)$$