1999
Boston Area Undergraduate
Physics Competition

Solutions

1. (a) Let the sphere have radius R and charge Q. Then the potential at the surface is

$$V(R) = \frac{Q}{R}. \tag{1}$$

The magnitude of the field at radius r inside the sphere is (from Gauss' law)

$$E(r) = \frac{Q(r^3/R^3)}{r^2} = \frac{Qr}{R^3}. \tag{2}$$

Integrating this from $r = R$ down to $r = 0$ gives a change in potential of $\Delta V = Q/2R$. Therefore, the potential at the center is

$$V(0) = \frac{Q}{R} + \frac{Q}{2R} = \frac{3Q}{2R}, \tag{3}$$

and the desired ratio is

$$\frac{V(R)}{V(0)} = \frac{2}{3}. \tag{4}$$

(b) Let ρ be the charge density of the cube. Let V_ℓ^{cor} be the potential at the corner of a cube of side ℓ. Let V_ℓ^{cen} be the potential at the center of a cube of side ℓ. By dimensional analysis,

$$V_\ell^{\text{cor}} \propto \frac{Q}{\ell} = \rho\ell^2. \tag{5}$$

Therefore,[1]

$$V_\ell^{\text{cor}} = 4V_{\ell/2}^{\text{cor}}. \tag{6}$$

But by superposition, we have

$$V_\ell^{\text{cen}} = 8V_{\ell/2}^{\text{cor}}, \tag{7}$$

because the center of the larger cube lies at a corner of the eight smaller cubes of which it is made. Therefore,

$$\frac{V_\ell^{\text{cor}}}{V_\ell^{\text{cen}}} = \frac{4V_{\ell/2}^{\text{cor}}}{8V_{\ell/2}^{\text{cor}}} = \frac{1}{2}. \tag{8}$$

2. The contact point on the ground does not look blurred, because it is instantaneously at rest. However, although this is the only point on the wheel that is at rest, there will be other locations in the picture where the spokes do not appear blurred.

The characteristic of a point in the picture where a spoke does not appear blurred is that the point lies on the spoke during the entire duration of the camera's exposure. (The point

[1] In other words, imagine expanding a cube with side $\ell/2$ to one with side ℓ. If we consider corresponding pieces of the two cubes, then the larger piece has $2^3 = 8$ times the charge of the smaller. But corresponding distances are twice as big in the large cube as in the small cube. Therefore, the larger piece contributes $8/2 = 4$ times as much to V_ℓ^{cor} as the smaller piece contributes to $V_{\ell/2}^{\text{cor}}$.

need not, however, correspond to the same point on the spoke.) At a certain time, consider a spoke in the lower half of the wheel. A short time later, the spoke will have moved, but it will intersect its original position. The spoke will not appear blurred at this intersection point. We must therefore find the locus of these intersections.

Let R be the radius of the wheel. Consider a spoke that makes an angle θ with the vertical. Let the wheel roll through an angle $d\theta$; then the center moves a distance $R\,d\theta$. The spoke's motion is a combination of a translation through a distance $R\,d\theta$, and a rotation through an angle $d\theta$ (about its top end).

Let r be the radial position of the intersection of the initial and final positions of the spoke. Then from the figure we have

$$(R\,d\theta)\cos\theta = r\,d\theta. \tag{9}$$

Therefore, $r = R\cos\theta$. This is easily seen to describe a circle whose diameter is the (lower) vertical radius of the wheel.

3. **First solution (slick method):**

Let β be the angle the force vector makes with the tangential direction. Let F be the maximum possible magnitude of the force of friction (it happens to be $F = \mu mg$, but we won't need this). The minimum-distance scenario is obtained when $F\sin\beta$ accounts for the radial acceleration, and the remaining $F\cos\beta$ accounts for the tangential acceleration. In other words,

$$F\sin\beta = \frac{mv^2}{R}, \qquad \text{and} \qquad F\cos\beta = m\dot{v}. \tag{10}$$

Taking the derivative of the first equation gives $F\cos\beta\dot{\beta} = 2mv\dot{v}/R$. Dividing this by the second equation gives $\dot{\beta} = 2v/R$. But $v = R\dot{\theta}$, where θ is the angular distance traveled around the circle. Therefore, $\dot{\beta} = 2\dot{\theta}$, and integration gives

$$\beta = 2\theta. \tag{11}$$

When the maximum speed is achieved, the value of β must be $\pi/2$. This value corresponds to

$$\theta = \frac{\pi}{4}. \tag{12}$$

Hence, the motorcycle must travel a distance $\pi R/4$, or one-eighth of the way around the circle.

Second solution (straightforward method):

In the minimum-distance scenario, the magnitude of the total force must be its maximum possible value, namely μmg (the exact form of this is not important). Since the radial force is $F_r = mv^2/R$, the tangential $F = ma$ equation is

$$F_t = \sqrt{(\mu mg)^2 - \left(\frac{mv^2}{R}\right)^2} = m\frac{dv}{dt}. \tag{13}$$

Multiplying through by dx, and then rewriting dx/dt as v (where $dx = R\,d\theta$ is the distance along the circle), we obtain[2]

$$dx = \frac{v\,dv}{\sqrt{(\mu g)^2 - \left(\frac{v^2}{R}\right)^2}}. \tag{14}$$

[2] This is simply the work-energy result, because the work is $F_t\,dx$, and the change in kinetic energy is $d(mv^2/2) = mv\,dv$.

Letting $z \equiv v^2/\mu g R$ gives

$$dx = \frac{R\,dz}{2\sqrt{1-z^2}}\,. \tag{15}$$

The maximum allowable speed, V, is obtained from $\mu m g = mV^2/R$. Therefore, $V^2 = \mu g R$, and the corresponding value of z is 1. The desired distance, X, is then

$$\begin{aligned} X = \int_0^X dx &= \int_0^1 \frac{R\,dz}{2\sqrt{1-z^2}} \\ &= \frac{R}{2}\int_0^{\pi/2} d\theta \qquad \text{(letting } z = \sin\theta) \\ &= \frac{\pi R}{4}\,. \end{aligned} \tag{16}$$

4. (a) The relative speed of the fox and rabbit, along the line connecting them, is always $v_{\text{rel}} = v - v\cos\alpha$. Therefore, the total time needed to decrease their separation from ℓ to zero is

$$T = \frac{\ell}{v(1-\cos\alpha)}\,. \tag{17}$$

This is valid unless $\alpha = 0$, in which case the fox never catches the rabbit.

The location of their meeting is a little trickier to obtain. We offer two methods.

First solution (slick method):

Imagine that the rabbit chases another rabbit, which chases another rabbit, etc. Each animal runs at an angle α relative to the direction directly away from the animal chasing it. The initial positions of all the animals lie on a circle, which is easily seen to have radius

$$R = \frac{\ell/2}{\sin(\alpha/2)}\,. \tag{18}$$

The center of the circle is the point, P, which is the vertex of the isosceles triangle with vertex angle α, and with the initial fox and rabbit positions as the other two vertices. By symmetry, the positions of the animals at all times must lie on a circle with center P. Therefore, P is the desired point where they meet. The hypothetical animals simply spiral in to P.

REMARK: An equivalent solution is the following. At all times, the rabbit's velocity vector is obtained by rotating the fox's velocity vector by α. In integrated form, the previous sentence says that the rabbit's net displacement vector is obtained by rotating the fox's net displacement vector by α. The meeting point, P, is therefore the vertex of the above-mentioned isosceles triangle.

Second solution (messier method):

This solution is a little messy, and not too enlightening, so we won't include every detail.

The speed of the rabbit in the direction orthogonal to the line connecting the two animals is $v\sin\alpha$. Therefore, during a time dt, the direction of the fox's motion changes by an angle $d\theta = v\sin\alpha\,dt/\ell_t$, where ℓ_t is the separation at time t. Hence, the change in the fox's velocity has magnitude $|d\mathbf{v}| = v\,d\theta = v(v\sin\alpha\,dt/\ell_t)$. The vector $d\mathbf{v}$ is orthogonal to $\mathbf{v}$; therefore, to get the x-component of $d\mathbf{v}$, we need to multiply

$|d\mathbf{v}|$ by v_y/v. Similar reasoning holds for the y-component of $d\mathbf{v}$, so we arrive at the two equations,

$$\begin{aligned} \dot{v}_x &= \frac{v v_y \sin\alpha}{\ell_t}, \\ \dot{v}_y &= -\frac{v v_x \sin\alpha}{\ell_t}. \end{aligned} \tag{19}$$

Now, we know that $\ell_t = (\ell - v(1-\cos\alpha)t)$. Multiplying the above two equations by ℓ_t, and integrating from the initial to final times (the left sides require integration by parts), yields

$$\begin{aligned} v_{x,0}\ell + v(1-\cos\alpha)X &= v\sin\alpha Y, \\ v_{y,0}\ell + v(1-\cos\alpha)Y &= -v\sin\alpha X, \end{aligned} \tag{20}$$

where (X, Y) is the total displacement vector, and $(v_{x,0}, v_{x,0})$ is the initial velocity vector. Putting all the X and Y terms on the right sides, and squaring and adding the equations, gives

$$\ell^2 v^2 = (X^2+Y^2)(v^2\sin^2\alpha + v^2(1-\cos\alpha)^2). \tag{21}$$

Therefore, the net displacement is

$$R = \sqrt{X^2+Y^2} = \frac{\ell}{\sqrt{2(1-\cos\alpha)}} = \frac{\ell/2}{\sin(\alpha/2)}. \tag{22}$$

To find the exact location, we can, with out loss of generality, set $v_{x,0} = 0$, in which case we find $Y/X = (1-\cos\alpha)/\sin\alpha = \tan(\alpha/2)$. This agrees with the result of the first solution.

(b) **First solution (slick method):**

Let $A(t)$ and $B(t)$ be the positions of the fox and rabbit, respectively. Let $C(t)$ be the foot of the perpendicular dropped from A to the line of the rabbit's path. Let α_t be the angle, as a function of time, at which the rabbit moves relative to the direction directly away from the fox (so $\alpha_0 \equiv \alpha$, and $\alpha_\infty = 0$).

The speed at which the distance AB decreases is equal to $v - v\cos\alpha_t$. And the speed at which the distance CB increases is equal to $v - v\cos\alpha_t$. Therefore, the sum of the distances AB and CB does not change. Initially, the sum is $\ell + \ell\cos\alpha$. In the end, it is $2d$, where d is the desired eventual separation. Therefore,

$$d = \frac{\ell(1+\cos\alpha)}{2}. \tag{23}$$

Second solution (straightforward method):

Let α_t be defined as in the first solution, and let ℓ_t be the separation at time t. The speed of the rabbit in the direction orthogonal to the line connecting the two animals is $v\sin\alpha_t$. The separation is ℓ_t, so the angle α_t changes at a rate

$$\dot{\alpha}_t = -\frac{v\sin\alpha_t}{\ell_t}. \tag{24}$$

And ℓ_t changes at a rate

$$\dot{\ell}_t = -v(1-\cos\alpha_t). \tag{25}$$

Taking the quotient of the above two equations, and separating variables, gives

$$\frac{\dot{\ell}_t}{\ell_t} = \frac{\dot{\alpha}_t(1 - \cos\alpha_t)}{\sin\alpha_t}. \tag{26}$$

The right side may be rewritten as $\dot{\alpha}_t \sin\alpha_t/(1 + \cos\alpha_t)$, and so integration gives

$$\ln(\ell_t) = -\ln(1 + \cos\alpha_t) + C, \tag{27}$$

where C is a constant of integration. Exponentiating gives $\ell_t(1 + \cos\alpha_t) = B$. The initial conditions demand that $B = \ell_0(1 + \cos\alpha_0) \equiv \ell(1 + \cos\alpha)$. Therefore,

$$\ell_t = \frac{\ell(1 + \cos\alpha)}{(1 + \cos\alpha_t)}. \tag{28}$$

Setting $t = \infty$, and using $\alpha_\infty = 0$, gives the final result

$$\ell_\infty = \frac{\ell(1 + \cos\alpha)}{2}. \tag{29}$$

REMARK: The solution of part b) is valid for all α except $\alpha{=}\pi$. If $\alpha{=}\pi$, the rabbit runs directly towards fox and they will indeed meet halfway in time $\ell/2v$.

5. Let ρ be the mass density of the raindrop, and let λ be the average mass density in space of the water droplets. Let $r(t)$, $M(t)$, and $v(t)$ be the radius, mass, and speed of the raindrop, respectively.

   The mass of the raindrop is $M = (4/3)\pi r^3\rho$. Therefore,

$$\dot{M} = 4\pi r^2\dot{r}\rho = 3M\frac{\dot{r}}{r}. \tag{30}$$

   Another expression for $\dot{M}$ is obtained by noting that the change in M is due to the acquisition of water droplets. The raindrop sweeps out volume at a rate given by its cross-sectional area times its velocity. Therefore,

$$\dot{M} = \pi r^2 v\lambda. \tag{31}$$

   The force of Mg on the droplet equals the rate of change of its momentum, namely $dp/dt = d(Mv)/dt = \dot{M}v + M\dot{v}$. Therefore,

$$Mg = \dot{M}v + M\dot{v}. \tag{32}$$

   We now have three equations involving the three unknowns, r, M, and v.

   (**Note:** We *cannot* write down the naive conservation-of-energy equation, because mechanical energy is *not* conserved. The collisions between the raindrop and the droplets are completely inelastic. The raindrop will, in fact, heat up. See the remark at the end of the solution.)

   The goal is to find $\dot{v}$ for large t. We will do this by first finding $\ddot{r}$ at large t. Eqs. (30) and (31) give

$$v = \frac{4\rho}{\lambda}\dot{r} \qquad \Longrightarrow \qquad \dot{v} = \frac{4\rho}{\lambda}\ddot{r}. \tag{33}$$

Plugging eqs. (30) and (33) into eq. (32) gives

$$Mg = \left(3M\frac{\dot{r}}{r}\right)\left(\frac{4\rho}{\lambda}\dot{r}\right) + M\left(\frac{4\rho}{\lambda}\ddot{r}\right). \tag{34}$$

Therefore,

$$\frac{g\lambda}{\rho}r = 12\dot{r}^2 + 4r\ddot{r}. \tag{35}$$

Given that the raindrop falls with constant acceleration at large times, we may write [3]

$$\ddot{r} \approx bg, \qquad \dot{r} \approx bgt, \qquad \text{and} \qquad r \approx \frac{1}{2}bgt^2, \tag{36}$$

for large t, where b is a numerical factor to be determined. Plugging eqs. (36) into eq. (35) gives

$$\left(\frac{g\lambda}{\rho}\right)\left(\frac{1}{2}bgt^2\right) = 12(bgt)^2 + 4\left(\frac{1}{2}bgt^2\right)bg. \tag{37}$$

Therefore, $b = \lambda/28\rho$. Hence, $\ddot{r} = g\lambda/28\rho$, and eq. (33) gives the acceleration of the raindrop at large t,

$$\dot{v} = \frac{g}{7}, \tag{38}$$

independent of ρ and λ.

Remark: We can calculate how much mechanical energy is lost (and therefore how much the raindrop heats up) as a function of the height fallen.

The fact that v is proportional to $\dot{r}$ (shown in eq. (33)) means that the volume swept out by the raindrop is a cone. The center-of-mass of a cone is 1/4 of the way from the base to the apex. Therefore, if M is the mass of the raindrop after it has fallen a height h, then the loss in mechanical energy is

$$E_{\text{lost}} = Mg\frac{h}{4} - \frac{1}{2}Mv^2. \tag{39}$$

Using $v^2 = 2(g/7)h$, this becomes

$$\Delta E_{\text{int}} = E_{\text{lost}} = \frac{3}{28}Mgh, \tag{40}$$

where ΔE_{int} is the gain in internal thermal energy. The energy required to heat 1g of water by 1 degree is 1 calorie (= 4.18 Joules). Therefore, the energy required to heat 1 kg of water by 1 degree is ≈ 4200 J. In other words,

$$\Delta E_{\text{int}} = 4200\, M\, \Delta T, \tag{41}$$

where mks units are used, and T is measured in celsius. (We have assumed that the internal energy is uniformly distributed throughout the raindrop.) Eqs. (40) and (41) give the increase in temperature as a function of h,

$$4200\, \Delta T = \frac{3}{28}gh. \tag{42}$$

How far must the raindrop fall before it starts to boil? If we assume that the water droplets' temperature is near freezing, then the height through which the raindrop must fall to have $\Delta T = 100\,^\circ\text{C}$ is found to be

$$h = 400\,\text{km}. \tag{43}$$

[3] We may justify the constant-acceleration statement in the following way. For large t, let r be proportional to t^α. Then the left side of eq. (35) goes like t^α, while the right side goes like $t^{2\alpha-2}$. If these are to be equal, then we must have $\alpha = 2$. Hence, $r \propto t^2$, and $\ddot{r}$ is a constant (for large t).

We have, of course, idealized the problem. But needless to say, there is no need to worry about getting burned by the rain.

A typical value for h is 10 km, which would raise the temperature by two or three degrees. This effect, of course, is washed out by many other factors.

6. Let $\theta(t)$ be the angle through which the spring moves. Let $x(t)$ be the length of the unwrapped part of the spring. Let $v(t)$ be the speed of the mass. And let $k(t)$ be the spring constant of the unwrapped part of the spring. (The manner in which k changes will be derived below.)

Using the approximation $a \ll L$, we may say that the mass undergoes approximate circular motion. (This approximation will break down when x becomes of order a, but the time during which this is true is negligible compared to the total time.) The instantaneous center of the circle is the point where the spring touches the pole. $F = ma$ along the instantaneous radial direction gives

$$\frac{mv^2}{x} = kx. \tag{44}$$

Using this value of v, the frequency of the circular motion is given by

$$\omega \equiv \frac{d\theta}{dt} = \frac{v}{x} = \sqrt{\frac{k}{m}}. \tag{45}$$

The spring constant, $k(t)$, of the unwrapped part of the spring is inversely proportional to its equilibrium length. (For example, if you cut a spring in half, the resulting springs have twice the original spring constant). All equilibrium lengths in this problem are infinitesimally small (compared to L), but the inverse relation between k and equilibrium length still holds. (If you want, you can think of the equilibrium length as a measure of the total number of spring atoms that remain in the unwrapped part.)

Note that the change in angle of the contact point on the pole equals the change in angle of the mass around the pole (which is θ.) Consider a small interval of time during which the unwrapped part of the spring stretches a small amount and moves through an angle $d\theta$. Then a length $a\,d\theta$ becomes wrapped on the pole. So the fractional decrease in the equilibrium length of the unwrapped part is (to first order in $d\theta$) equal to $(a\,d\theta)/x$. From the above paragraph, the new spring constant is therefore

$$k_{\text{new}} = \frac{k_{\text{old}}}{1 - \frac{a\,d\theta}{x}} \approx k_{\text{old}}\left(1 + \frac{a\,d\theta}{x}\right). \tag{46}$$

Therefore, $dk = ka\,d\theta/x$. Dividing by dt gives

$$\dot{k} = \frac{ka\omega}{x}. \tag{47}$$

The final equation we need is the one for energy conservation. At a given instant, consider the sum of the kinetic energy of the mass, and the potential energy of the unwrapped part of the spring. At a time dt later, a tiny bit of this energy will be stored in the newly-wrapped little piece. Letting primes denote quantities at this later time, conservation of energy gives

$$\frac{1}{2}kx^2 + \frac{1}{2}mv^2 = \frac{1}{2}k'x'^2 + \frac{1}{2}m'v'^2 + \left(\frac{1}{2}kx^2\right)\left(\frac{a\,d\theta}{x}\right). \tag{48}$$

The last term is (to lowest order in $d\theta$) the energy stored in the newly-wrapped part, because $a\,d\theta$ is its length. Using eq. (44) to write the v's in terms of the x's, this becomes

$$kx^2 = k'x'^2 + \frac{1}{2}kxa\,d\theta. \tag{49}$$

In other words, $-(1/2)kxa\,d\theta = d(kx^2)$. Dividing by dt gives

$$\begin{aligned} -\frac{1}{2}kxa\omega &= \frac{d(kx^2)}{dt} \\ &= \dot{k}x^2 + 2kx\dot{x} \\ &= \left(\frac{ka\omega}{x}\right)x^2 + 2kx\dot{x}, \end{aligned} \tag{50}$$

where we have used eq. (47). Therefore,

$$\dot{x} = -\frac{3}{4}a\omega. \tag{51}$$

We must now solve the two couple differential equations, eqs. (47) and (51). Dividing the latter by the former gives

$$\frac{\dot{x}}{x} = -\frac{3}{4}\frac{\dot{k}}{k}. \tag{52}$$

Integrating and exponentiating gives

$$k = \frac{KL^{4/3}}{x^{4/3}}, \tag{53}$$

where the numerator is obtained from the initial conditions, $k = K$ and $x = L$. Plugging eq. (53) into eq. (51), and using $\omega = \sqrt{k/m}$, gives

$$x^{2/3}\dot{x} = -\frac{3aK^{1/2}L^{2/3}}{4m^{1/2}}. \tag{54}$$

Integrating, and using the initial condition $x = L$, gives

$$x^{5/3} = L^{5/3} - \left(\frac{5aK^{1/2}L^{2/3}}{4m^{1/2}}\right)t. \tag{55}$$

So, finally,

$$x(t) = L\left(1 - \frac{t}{T}\right)^{3/5}, \tag{56}$$

where

$$T = \frac{4}{5}\frac{L}{a}\sqrt{\frac{m}{K}} \tag{57}$$

is the time for which $x(t) = 0$ and the mass hits the pole.

REMARKS:

(a) Note that the angular momentum of the mass around the center of the pole is *not* conserved in this problem, because the force is not a central force.

(b) Integrating eq. (51) up to the point when the mass hit the pole gives $-L = -(3/4)a\theta$. But $a\theta$ is the total length wrapped around the pole, which we see is equal to $4L/3$.