1997
The Boston Area Undergraduate Physics Competition

Solutions

1. (a) In case of the air bubble the level will stay the same, since the volume of the displaced water is exactly equal to the "extra" volume of water that came from the molten block.

   In case of the frozen in piece of lead, the level of water will decrease, because the volume of the water displaced due to the lead piece exceeds the volume of the piece.

   (b) Since the positrons are a lot lighter than the protons, it makes sense to divide the problem in two parts.

   First, the positrons fly away while the protons stay in place. The energy conservation requires that

$$4 \times \frac{1}{2}\alpha(\frac{1}{l} + \frac{1}{l} + \frac{1}{\sqrt{2}l}) = \frac{\alpha}{\sqrt{2}l} + 2\frac{mv_{e^+}^2}{2}, \tag{1}$$

   where $\alpha = 2.3 \times 10^{-28} Jm$, or

$$v_{e^+} = \sqrt{\frac{\alpha}{ml}(4 + 1/\sqrt{2})} \approx 1100 m/s \tag{2}$$

   After that, the protons fly away:

$$v_p = \sqrt{\alpha/\sqrt{2}Ml} \approx 9.8 m/s. \tag{3}$$

2. The forces on the particle are the friction force and the magnetic force, so $\vec{F} = -\alpha\vec{v} + q\vec{v} \times \vec{B}$. The particle will remain in the x-y plane, so $\vec{v}$ has no component in the $\hat{z}$ direction. The cross product takes a simple form, and we have

$$\begin{aligned} F_x &= -\alpha v_x + qBv_y, \\ F_y &= -\alpha v_y - qBv_x. \end{aligned} \tag{4}$$

   Equivalently,

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\alpha\frac{dx}{dt} + qB\frac{dy}{dt}, \\ m\frac{d^2y}{dt^2} &= -\alpha\frac{dy}{dt} - qB\frac{dx}{dt}. \end{aligned} \tag{5}$$

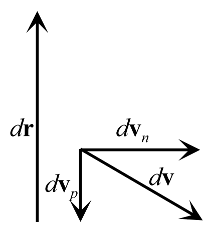
Integrating these equations from the time the particle enters $\mathcal{R}_{y>0}$ to the time it comes to rest, we obtain

$$\begin{aligned} m\Delta v_x &= -\alpha\Delta x + qB\Delta y, \\ m\Delta v_y &= -\alpha\Delta y - qB\Delta x, \end{aligned} \tag{6}$$

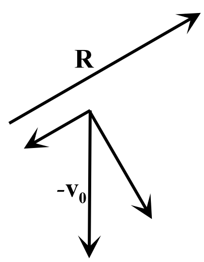
We want to find the coordinates of P, namely $(\Delta x, \Delta y)$.

The initial velocity is $\vec{v} = (0, v_0, 0)$, and the final velocity is $\vec{v} = (0, 0, 0)$, so $\Delta v_x = 0$ and $\Delta v_y = -v_0$. Solving eqs. (6) for Δx and Δy gives

$$\Delta x = \frac{qBmv_0}{\alpha^2 + (qB)^2}, \qquad \Delta y = \frac{\alpha m v_0}{\alpha^2 + (qB)^2}. \tag{7}$$



There is an equivalent geometrical solution. Consider a small interval of time dt. The particle will change its position by $d\mathbf{r} = \mathbf{v}dt$. Over the same time period, the speed will change by $d\mathbf{v}_p = \frac{\alpha}{m}vdt$ opposite to the direction of motion, and by $d\mathbf{v}_n = \frac{qB}{m}vdt$ in the direction normal to it. Note that the components of the velocity change are proportional to the displacement.



Similar set of vectors can be drawn for any small dt. Summing them up, and knowing the total change in the particle speed $(-\mathbf{v}_0)$, we can draw the final diagram. Since the proportionality is conserved in vector addition, the total displacement $\mathbf{R}$ (and, of course, its components) can be easily found.

3. The key to this problem is to realize that the stick will lose contact with the wall before it hits the ground. The first thing we must do is calculate exactly where this loss of contact occurs.

   Let $r = \ell/2$, for convenience. It is easy to see that while the stick is in contact with the wall, the center of mass of the stick will move in a circle of radius r. Let θ be the angle between the wall and the radius from the corner to the CM of the stick. (This is also the angle between the stick and the wall.)

   We will solve the problem by assuming that the CM always moves in a circle, and then determining the point at which the horizontal CM speed starts to decrease (i.e., the point at which the normal force from the wall becomes negative [which it of course can't do]).

By conservation of energy, the kinetic energy of the stick is equal to the loss in potential energy, which is $mgr(1-\cos\theta)$, where θ is defined above. This kinetic energy may be broken up into the CM translational energy plus the rotation energy. The CM translational energy is simply $mr^2\dot{\theta}^2/2$ (since the CM travels in a circle). The rotational energy is $I\dot{\theta}^2/2$. (The same $\dot{\theta}$ applies here as in the CM translational motion, because θ is the angle between the stick and the vertical.) Letting $I \equiv \rho mr^2$, to be general ($\rho = 1/3$ for our stick), we have, by conservation of energy, $(1+\rho)mr^2\dot{\theta}^2/2 = mgr(1-\cos\theta)$. Therefore, the speed of the CM, $v = r\dot{\theta}$, is

$$v = \sqrt{\frac{2gr}{1+\rho}}\sqrt{(1-\cos\theta)}\,. \tag{8}$$

The horizontal speed is therefore

$$v_x = \sqrt{\frac{2gr}{1+\rho}}\sqrt{(1-\cos\theta)}\cos\theta. \tag{9}$$

Taking the derivative of $\sqrt{(1-\cos\theta)}\cos\theta$, we see that the speed is maximum at $\cos\theta = 2/3$. (This is independent of ρ.)

Therefore the stick loses contact with the wall when

$$\cos\theta = 2/3. \tag{10}$$

Using this value of θ in eq. (9) gives a horizontal speed of (letting $\rho = 1/3$)

$$v_x = \frac{1}{3}\sqrt{2gr} = \frac{1}{3}\sqrt{gl}. \tag{11}$$

This is the horizontal speed just after the stick loses contact with the wall, and thus is the horizontal speed from then on, because the floor exerts no horizontal force.

4. Consider the collision between two sticks. Let the speed of the end of the heavy one be V. Since this stick is essentially infinitely heavy, we may consider it to be an infinitely heavy ball, moving at speed V. (The translational degree of freedom of the heavy stick is irrelevant, as far as the light stick is concerned.)

   In the same spirit as the (easier) problem of the collision between two balls of greatly disparate masses, we will work out this problem in the rest frame of the infinitely heavy ball right before the collision. (The problem can be done in the lab frame, but our method here is a little less messy.) The situation reduces to a stick of mass m, length $2r$, moment of inertia ρmr^2, and speed V, approaching a fixed wall To find the behavior of the stick after the collision, we will use (1) conservation of energy, and (2) conservation of angular momentum around the contact point.

Let u be the speed of the center of mass of the stick after the collision. Let ω be its angular velocity after the collision.

Since the wall is infinitely heavy, it will acquire zero kinetic energy. So conservation of E gives

$$\frac{1}{2}mV^2 = \frac{1}{2}mu^2 + \frac{1}{2}(\rho mr^2)\omega^2. \tag{12}$$

The initial angular momentum around the contact point is $L = mrV$, so conservation of L gives (breaking L after the collision up into the L of the CM plus the L relative to the CM)

$$mrV = mru + (\rho mr^2)\omega. \tag{13}$$

Solving eqs. (12) and (13) for u and $r\omega$ in terms of V gives

$$u = V\frac{1-\rho}{1+\rho}, \qquad \text{and} \qquad rw = V\frac{2}{1+\rho}. \tag{14}$$

(The other solution, $u = V$ and $r\omega = 0$ represents the case where the stick misses the wall.)

Going back to the lab frame (i.e., subtracting V from the speed u) we see that the collision gives the the lighter stick a CM speed equal to $v = 2V\rho/(1+\rho)$ in the same direction as the original V. But the far end of the light stick has a backwards rotational speed equal to $r\omega = 2V/(1+\rho)$. This rotational speed is greater than the CM speed, so the far end of the light stick travels at a speed

$$V' = r\omega - v = V\frac{2(1-\rho)}{1+\rho} \tag{15}$$

in the direction opposite to the original V.

The same analysis works in the next collision. In other words, the bottom ends of the sticks move with speeds that form a geometric progression with ratio $2(1-\rho)/(1+\rho)$. If this ratio is less than 1 (i.e., $\rho > 1/3$), then the speeds go to zero, as $n \to \infty$. If it is greater than 1 (i.e., $\rho < 1/3$), then the speeds go to infinity, as $n \to \infty$. If it equals 1 (i.e., $\rho = 1/3$), then the speeds are independent of n, as $n \to \infty$.

Therefore,

$$\rho = \frac{1}{3} \tag{16}$$

is the desired answer.

A uniform stick has $\rho = 1/3$ (usually written in the form $I = m\ell^2/12$, where $\ell = 2r$).

Although in the case where ρ is strictly equal to zero the centers of masses of the sticks would remain at rest while their rotational speeds would form a geometric progression with ratio 2, we are not considering this to be the correct answer. Real objects with finite mass have a finite moment of inertia. A proper physical approach then would require taking a limit as $\rho \to 0$, but in that case the limit as $n \to \infty$ is infinite.

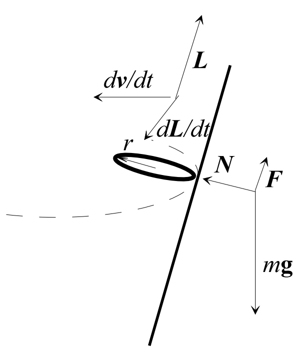
5. (a) The forces on the particle are gravity (mg) and the normal force (N) from the cone. In our situation, there is no net force in the vertical direction, so

$$N \sin\theta = mg, \tag{17}$$

i.e., $N = mg/\sin\theta$. Therefore, the inward horizontal force, $N\cos\theta$, equals $mg/\tan\theta$. This force must account for the centripetal acceleration of the particle moving in a circle of radius $h\tan\theta$. Hence, $mg/\tan\theta = m(h\tan\theta)\omega^2$, and

$$\omega = \sqrt{\frac{g}{h}\frac{1}{\tan\theta}}. \tag{18}$$

(b)



The forces on the ring are gravity (mg), the normal force (N) from the cone, and a friction force (F) pointing up along the cone. In our situation, there is no net force in the vertical direction, so

$$N\sin\theta + F\cos\theta = mg. \tag{19}$$

The fact that the inward horizontal force accounts for the centripetal acceleration yields

$$N\cos\theta - F\sin\theta = m(h\tan\theta)\omega^2. \tag{20}$$

We must now consider the torque, $\vec{\tau}$, on the ring. The torque is due solely to F (because gravity provides no torque, and N points though the center of the ring, by assumption (2) in the problem). So

$$\tau = rF, \tag{21}$$

and it points in the direction along the circular motion. Since $\vec{\tau} = d\vec{L}/dt$, we must now find $d\vec{L}/dt$.

$\vec{L}$ is made up of two pieces. One comes from the center of mass motion of the ring, which revolves around the axis of the cone. This part of $\vec{L}$ does not change, so we may neglect it in calculating $d\vec{L}/dt$. The other piece comes from the rotation of the ring. Let us call this part $\vec{L}'$. It points up along the cone, so the $\vec{L}'$ vector traces out a cone in which the tip of $\vec{L}'$ moves in a circle of radius $L'\sin\theta$. The frequency of this circular motion is of course the same ω as above. Therefore,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}'}{dt} = \omega L' \sin\theta \tag{22}$$

(in the direction of the circular motion).
Thus, $\vec{\tau} = d\vec{L}/dt$ gives

$$rF = \omega L' \sin\theta. \tag{23}$$

But $L' = mr^2\omega'$, where ω' is the angular speed of the ring. And we know that ω' and ω are related by $r\omega' = (h\tan\theta)\omega$ (the rolling-without-slipping condition)[1]. Therefore $L' = mr(h\tan\theta)\omega$. Using this in eq. (23) yields

$$F = m\omega^2(h\tan\theta)\sin\theta. \tag{24}$$

Eqs. (19), (20), and (24) have the three unknowns, N, F, and ω. We can solve for ω by multiplying eq. (19) by $\cos\theta$, and eq. (20) by $\sin\theta$, and taking the difference, to obtain

$$F = mg\cos\theta - m\omega^2(h\tan\theta)\sin\theta. \tag{25}$$

Equating this expression for F with that in eq. (24) gives

$$\omega = \sqrt{\frac{g}{2h}\frac{1}{\tan\theta}}. \tag{26}$$

This frequency is $1/\sqrt{2}$ times the frequency found in part (a).
REMARK: If one considers an object with moment of inertia $\rho m r^2$ (our ring has $\rho = 1$), then one can show by the above reasoning that the "2" in eq. (26) is simply replaced by $(1+\rho)$.

6. (a) **First Solution:**

Consider one of the collisions of the block and the particle. Let V and v be the speeds of the block and particle, respectively, after the collision. Let the collision occur at a distance ℓ from the wall. We claim the the quantity $\ell(v-V)$ is an invariant (i.e., for each collision, it is the same). The proof is as follows.

[1] Actually, this isn't quite true, for the same reason that the earth spins around 366 instead of 365 times in a year. But it's valid enough, in the limit of small r.

Let us find the position, ℓ', of the following collision. The time to the next collision is given by $Vt + vt = 2\ell$ (because the sum of the distances traveled by the two objects is 2ℓ). Therefore, since $\ell' = \ell - Vt$, we have

$$\ell' = \ell(v - V)/(v + V). \tag{27}$$

We now invoke the handy fact that in an elastic collision, the relative speed of the particles before the collision is equal to the relative speed after the collision. (This is most easily proved by working in the center of mass frame.) Therefore, letting V' and v' be the speeds of the block and particle, respectively, after the next collision, we have

$$v + V = v' - V'. \tag{28}$$

Using this in eq. (27) gives

$$\ell'(v' - V') = \ell(v - V), \tag{29}$$

as was to be shown.

What is the value of this invariant? After the first collision, the block continues to move essentially with speed V_0 (up to corrections of order m/M), and the particle acquires a speed essentially equal to $2V_0$ (up to corrections of order m/M). (The latter is most easily seen by working in the frame of the heavy block.) So the invariant $\ell(v - V)$ is essentially equal to $L(2V_0 - V_0) = LV_0$.

Let L_c be the closest distance to the wall. When the block is at this closest point to the wall, its speed is zero. Therefore, all of the initial kinetic energy of the block belongs to the particle. Thus, $v = V_0\sqrt{M/m}$. So at this point our invariant tells us $LV_0 \approx L_c(V_0\sqrt{M/m} - 0)$, and so

$$L_c \approx L\sqrt{\frac{m}{M}}. \tag{30}$$

Second Solution:

Let $V(t)$ be the speed of the block, and let $v(t)$ be the speed of the particle. Let $x(t)$ be the distance from the wall to the block.

The block reaches its closest point to the wall when all of its initial kinetic energy is transferred to the particle, i.e.,

$$\frac{1}{2}mv^2 = \frac{1}{2}MV_0^2. \tag{31}$$

Hence, $v = V_0\sqrt{M/m}$ at this point. Therefore, if we can find a relation between $v(t)$ and $x(t)$, we are done.

We claim that for $M \gg m$, the product $v(t)x(t)$ is essentially equal to $V_0 L$. The proof is as follows.

Consider the later times when the bounces are very frequent, and when v is very large, so that a large number, dn, of bounces occur during a small period of time, dt, where x does not change significantly.

Since the particle travels a distance $v\,dt$ during a time dt, and since the distance from the block to the wall and back is $2x$, we have

$$dn = \frac{v\,dt}{2x}. \tag{32}$$

Each collision between the block and the particle increases the particle's speed by essentially $2V$ (because $M \gg m$, so the block is essentially infinitely heavy). Therefore, the increase in the speed of the particle during the time dt is

$$dv = 2V\,dn = \frac{V v\,dt}{x}. \tag{33}$$

But $V = -dx/dt$, so we have

$$dv = -\frac{v\,dx}{x}. \tag{34}$$

Dividing by v and integrating gives $\ln v = -\ln x + (\text{const})$. Therefore,

$$vx = C. \tag{35}$$

Thus, the speed of the particle is inversely proportional to the distance between the wall and the block.

This is just what we expect if we consider the particle to be a one-dimensional gas. Indeed, the adiabatic compression of such gas would be governed by an invariant $PV^\gamma = const$, where

$$\gamma = \frac{C_P}{C_V} = \frac{C_V + 1}{C_V} = \frac{\frac{i}{2} + 1}{\frac{i}{2}}, \tag{36}$$

where i is the number of degrees of freedom. For the one-dimensional gas, $i = 1$, so $\gamma = 3$ and $PV^3 = const$. Since PV is proportional to T, or v^2, V is x in the one-dimensional case, $v^2 x^2 = const$, or $vx = C$.

We must now determine the constant C by looking at the first few collisions.

The first collision gives the particle a speed $2V_0$ (it's not quite $2V_0$, but the error is of order m/M). After the collision, the block continues to move with (essentially) speed V_0, so it is easy to calculate that the second collision occurs at (roughly) $x = L/3$.

After the second collision, the particle has speed $4V_0$,[2] while the block continues to have speed V_0, so we find that the third collision occurs at $x = L/5$.

After the third collision, the particle has speed $6V_0$, while the block continues to have speed V_0, so we find that the fourth collision occurs at $x = L/7$.

In general, we find that the kth collision occurs at $L/(2k-1)$. And the particle had speed $2(k-1)V_0$ before the collision, and $2kV_0$ after it. This is valid up to order m/M corrections, as long as k is not too large.

For large enough M/m, we may make this realm overlap with the above realm where there is a large number of collisions in a short period of time. So we find $C = vx \approx (2kV_0)L/(2k-1) \approx V_0 L$, since k can be made large if M/m is very large. Therefore,

$$vx \approx V_0 L. \tag{37}$$

Using $v \approx V_0 L/x$ in eq. (31), we find that the closest approach, x_c, of the block to the wall is

$$x_c \approx L\sqrt{\frac{m}{M}}. \tag{38}$$

(b) **First Solution:**

Let V and v be the speeds of the block and particle, respectively.

The decrease in the momentum of the block due to a bounce is equal to the change in the momentum of the particle from the bounce, which is roughly equal to $2mv$ (we are assuming V small compared to v, which is the case after the first few collisions). If there are dn bounces in a time dt, then conservation of momentum during the time dt gives (assuming v stays fairly constant throughout the small interval dt).

$$MdV = -2mvdn. \tag{39}$$

Conservation of energy, $MV^2/2 + mv^2/2 = MV_0^2/2$, allows us to write v in terms of V:

$$v = V_0\sqrt{\frac{M}{m}}\sqrt{1-\frac{V^2}{V_0^2}}. \tag{40}$$

Eq. (39) then gives (changing variables to $y \equiv V/V_0$, and integrating up to the

[2]It should be understood that this and all the other numbers in the next few paragraphs are approximations which become arbitrarily accurate in the limit $M \gg m$.

closest approach to the wall, which corresponds to $V = 0$, and hence $y = 0$)

$$\frac{1}{2}\sqrt{\frac{M}{m}}\int_1^0 \frac{dy}{\sqrt{1-y^2}} = -\int_0^N dn = -N. \tag{41}$$

(The integrand is actually not valid for V near V_0, i.e. for y near 1, because eq. (39) is not valid there. But the error there is small compared to the total number of bounces.)

The integral gives $\arcsin y$, which yields $-\pi/2$ when evaluated between 1 and 0. So the total number of bounces is

$$N \approx \frac{\pi}{4}\sqrt{\frac{M}{m}}, \tag{42}$$

Second Solution:

Let V and v be the speeds of the block and particle, respectively, after a given collision. Let V' and u' be the speeds of the block and particle, respectively, after the following collision. Conservation of momentum in this second collision gives

$$MV - mv = MV' + mv'. \tag{43}$$

This equation, together with eq. (28), allows us to solve for V' and v' in terms of V and v. In matrix form, we obtain

$$\begin{pmatrix} V' \\ v' \end{pmatrix} = \begin{pmatrix} \frac{M-m}{M+m} & \frac{-2m}{M+m} \\ \frac{2M}{M+m} & \frac{M-m}{M+m} \end{pmatrix} \begin{pmatrix} V \\ v \end{pmatrix}. \tag{44}$$

The eigenvectors, A_i, and eigenvalues, λ_i, of this matrix are

$$A_1 = \begin{pmatrix} 1 \\ -i\sqrt{\frac{M}{m}} \end{pmatrix}, \quad \lambda_1 = \frac{M-m}{M+m} + \frac{2i\sqrt{Mm}}{M+m} \equiv e^{i\theta}, \tag{45}$$

$$A_2 = \begin{pmatrix} 1 \\ i\sqrt{\frac{M}{m}} \end{pmatrix}, \quad \lambda_2 = \frac{M-m}{M+m} - \frac{2i\sqrt{Mm}}{M+m} \equiv e^{-i\theta}, \tag{46}$$

where $\theta \equiv \arctan(2\sqrt{Mm}/(M+m)) \approx 2\sqrt{m/M}$.

The initial conditions are $(V, v) = (V_0, 0) = (V_0/2)(A_1 + A_2)$. Therefore, the speeds after the nth bounce are given by

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \frac{V_0}{2}(\lambda_1^n A_1 + \lambda_2^n A_2). \tag{47}$$

Writing $\lambda_1 = \epsilon^{i\theta}$, $\lambda_2 = \epsilon^{-i\theta}$, and using the explicite form of the A_i, we have

$$\begin{pmatrix} V_n \\ v_n \end{pmatrix} = \frac{V_0}{2}(\epsilon^{in\theta} A_1 + \epsilon^{-in\theta} A_2) = V_0 \begin{pmatrix} \cos n\theta \\ \sqrt{\frac{M}{m}} \sin n\theta \end{pmatrix}. \tag{48}$$

The block makes its closest approach to the wall when $V_N = 0$, i.e., when $N\theta = \pi/2$. Using the definition of θ gives

$$\begin{aligned} N &= \frac{\pi}{2} \frac{1}{\arctan \frac{2\sqrt{Mm}}{M+m}} \\ &\approx \frac{\pi}{4}\sqrt{\frac{M}{m}}. \end{aligned} \tag{49}$$