# Solutions

9th Annual
Boston Area Undergraduate
Physics Competition

April 26, 2003

1. At time $t$, the free end of the rope is moving at speed $gt$, and it has fallen a distance $gt^2/2$. This distance gets "doubled up" below the support. So at time $t$, a length $gt^2/4$ is hanging at rest, and a length $L - gt^2/4$ is moving at speed $gt$. The momentum of the entire rope is therefore $p = -\rho(L - gt^2/4)(gt)$, where $\rho \equiv M/L$ is the mass density, and the minus sign signifies downward motion.

   The forces on the entire rope are $Mg = \rho Lg$ downward, and $N$ (the force from the support) upward. $F = dp/dt$ applied to the entire rope therefore gives

$$\begin{aligned} N - \rho Lg &= \frac{d}{dt}\left(-\rho Lgt + \frac{\rho g^2 t^3}{4}\right) \\ \Longrightarrow \qquad N &= \frac{3\rho g^2 t^2}{4}\,. \end{aligned} \tag{1}$$

   (Note that this equals $3\rho v^2/4$.) This result holds until $t = \sqrt{4L/g}$, which is the time it takes the free end to fall a distance $2L$. After this time, the force from the support is simply the weight of the entire rope, $Mg$.

   REMARK: At time $t$, the part of the rope that is hanging at rest has weight $\rho(gt^2/4)g = \rho g^2 t^2/4$. From eq. (1), we see that the support must apply a force that is three times the weight of this motionless part of the rope. The extra force is necessary because the support must do more than hold up the motionless part. It must cause the change in momentum of the atoms in the rope that are abruptly brought to rest from their freefall motion.

2. Because the pole is very thin, we can approximate the motion of the mass at all times by a circle. Let $\theta$ be the desired angle of the string when it becomes completely unwound. Then the total change in height of the mass is $L\cos\theta$. So conservation of energy gives

$$mgL\cos\theta = \frac{mv^2}{2}\,. \tag{2}$$

   The vertical component of the tension in the string is essentially $mg$, because the height of the mass changes so slowly (because the pole is so thin). Therefore, the horizontal component of the tension is $mg\tan\theta$. The mass travels in a horizontal circle of radius $L\sin\theta$, so the horizontal $F = ma$ equation gives

$$mg\tan\theta = \frac{mv^2}{L\sin\theta}\,. \tag{3}$$

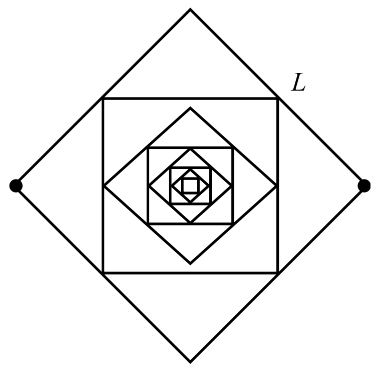
   Dividing eq. (3) by eq. (2) gives

$$\tan\theta = \sqrt{2}. \tag{4}$$

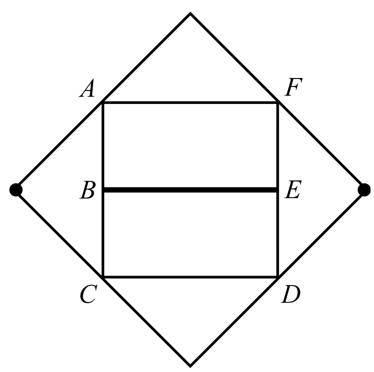
   The numerical value turns out to be $\theta \approx 54.7°$.

3. Let the desired resistance between opposite corners be $xR$, where $x$ is a numerical factor to be determined. Consider all the squares except the largest two. Label this subset of resistors as $S_3$. $S_3$ is identical to the original circuit, except that it is shrunk by a factor of 2. Therefore, the resistance between the right and left corners of $S_3$ is $xR/2$ (because all the cross sections are the same, and resistance is proportional to length).

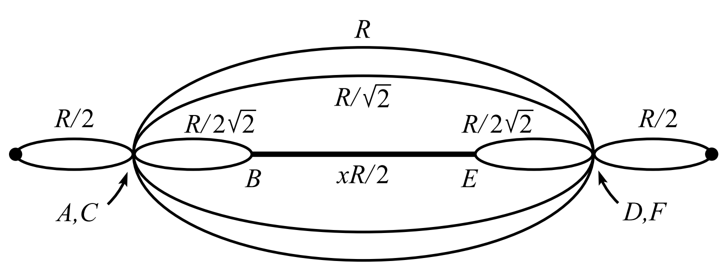
   From left-right symmetry, all the points on the vertical bisector of the circuit below are at the same potential. Therefore, we can separate the top and bottom corners of $S_3$ from the horizontal lines they touch, as shown.



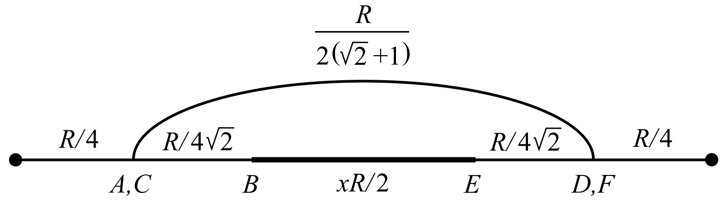

   We can then think of $S_3$ as an effective resistor of resistance $xR/2$, as shown.




   We can now simplify this circuit by noting the top-bottom symmetry, which tells us that we can identify $A$ with $C$, and also $D$ with $F$. (Note that we *cannot* identify $B$ with $A, C$; or $E$ with $D, F$.) We arrive at:



Reducing things a bit more gives:




Reducing one last time, and setting the result equal to $xR$, gives:

$$xR = \frac{R}{2} + \left( \frac{1}{\frac{R}{2(\sqrt{2}+1)}} + \frac{1}{\frac{R}{2\sqrt{2}} + \frac{xR}{2}} \right)^{-1} \tag{5}$$

We must now solve for $x$. Gradual simplification gives:

$$2x - 1 = \frac{1}{\sqrt{2} + 1 + \frac{\sqrt{2}}{\sqrt{2}x+1}}$$

$$\Longrightarrow \quad (2x-1)\left((2+\sqrt{2})x + 2\sqrt{2} + 1\right) = \sqrt{2}x + 1$$

$$\Longrightarrow \quad (2+\sqrt{2})x^2 + \sqrt{2}x - (\sqrt{2}+1) = 0$$

$$\Longrightarrow \quad x^2 + (\sqrt{2}-1)x - \frac{1}{\sqrt{2}} = 0$$

$$\Longrightarrow \quad x = \frac{1}{2}(\sqrt{3} - \sqrt{2} + 1) \approx 0.659. \tag{6}$$

4. Integrating $\mathbf{F} = d\mathbf{p}/dt$ gives

$$\Delta\mathbf{p} = \int \mathbf{F}\, dt. \tag{7}$$

Before the magnetic field is turned on, the force on the particle takes the form, $\mathbf{F} = -b\mathbf{v}$. Therefore,

$$\Delta\mathbf{p} = \int (-b\mathbf{v})\, dt. \tag{8}$$

But $\int \mathbf{v}\, dt = \Delta\mathbf{x}$, where $\Delta\mathbf{x}$ is the total displacement (which equals 10 cm straight into the region here). So we have

$$\Delta\mathbf{p}_0 = -b\Delta\mathbf{x}_0, \tag{9}$$

where the subscript denotes the zero-$\mathbf{B}$ case.

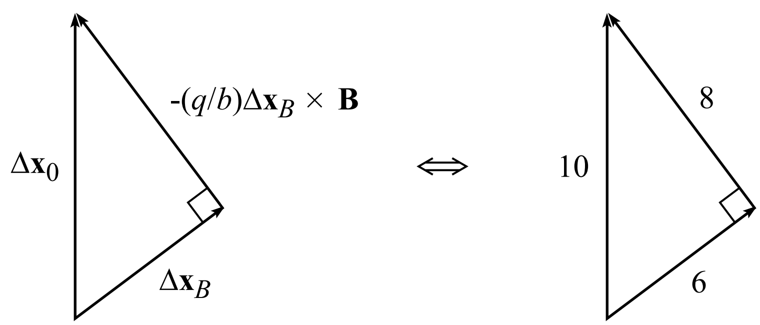
Let us now turn on the magnetic field. We have $\mathbf{F} = -b\mathbf{v} + q\mathbf{v}\times\mathbf{B}$, so

$$\begin{aligned} \Delta\mathbf{p} &= \int (-b\mathbf{v} + q\mathbf{v}\times\mathbf{B})\, dt \\ &= -b\Delta\mathbf{x} + q\Delta\mathbf{x}\times\mathbf{B}. \end{aligned} \tag{10}$$

But the $\Delta\mathbf{p}$ here is the same as the $\Delta\mathbf{p}_0$ in eq. (9), because the the particle has the same initial velocity and the same final velocity (namely zero). Therefore, we have (after dividing through by $-b$)

$$\Delta\mathbf{x}_0 = \Delta\mathbf{x}_B - (q/b)\Delta\mathbf{x}_B \times \mathbf{B}. \tag{11}$$

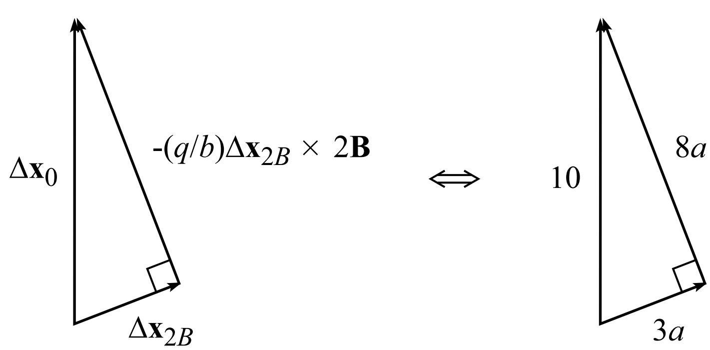
The two terms on the righthand side represent orthogonal vectors. Since the sum of these two orthogonal vectors equals the vector on the lefthand side, and since $|\Delta\mathbf{x}_B|/|\Delta\mathbf{x}_0| = 6/10$, we see that we have the following 6-8-10 right triangle.




If we now double the magnetic field, we have

$$\Delta\mathbf{x}_0 = \Delta\mathbf{x}_{2B} - (q/b)\Delta\mathbf{x}_{2B} \times 2\mathbf{B}. \tag{12}$$

The ratio of the magnitudes of the two vectors on the righthand side is twice the ratio of the two vectors in eq. (11). That is, it is 8/3 instead of 8/6. So we now have the following right triangle.




The Pythagorean theorem gives the value of $a$ as $10/\sqrt{73}$. Therefore, the net displacement of the particle is $|\Delta\mathbf{x}_{2B}| = 3a = 30/\sqrt{73} \approx 3.51\,\text{cm}$.

5. Let us ignore the tilt of the plane for a moment and determine how the $\omega_f$ and $v_f$ after a bounce are related to the $\omega_i$ and $v_i$ before the bounce (where $v$ denotes the velocity component parallel to the plane). Let the positive directions of velocity and force be to the right along the plane, and let the positive direction of angular velocity

be counterclockwise. If we integrate (over the small time of a bounce) the friction force and the resulting torque, we obtain

$$\begin{aligned} F &= \frac{dp}{dt} \qquad \Longrightarrow \qquad \int F\,dt = \Delta p, \\ \tau &= \frac{dL}{dt} \qquad \Longrightarrow \qquad \int \tau\,dt = \Delta L. \end{aligned} \tag{13}$$

But $\tau = RF$. And since $R$ is constant, we have

$$\Delta L = \int RF\,dt = R\int F\,dt = R\Delta p. \tag{14}$$

Therefore,

$$I(\omega_f - \omega_i) = Rm(v_f - v_i). \tag{15}$$

But conservation of energy gives

$$\begin{aligned} \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega_f^2 &= \frac{1}{2}mv_i^2 + \frac{1}{2}I\omega_i^2 \\ \Longrightarrow \qquad I(\omega_f^2 - \omega_i^2) &= m(v_i^2 - v_f^2). \end{aligned} \tag{16}$$

Dividing this equation by eq. (15) gives[1]

$$R(\omega_f + \omega_i) = -(v_f + v_i). \tag{17}$$

We can now combine this equation with eq. (15), which can be rewritten (using $I = (2/5)mR^2$) as

$$\frac{2}{5}R(\omega_f - \omega_i) = v_f - v_i. \tag{18}$$

Given $v_i$ and $\omega_i$, the previous two equations are two linear equations in the two unknowns, $v_f$ and $\omega_f$. Solving for $v_f$ and $\omega_f$, and then writing the result in matrix notation, gives

$$\begin{pmatrix} v_f \\ R\omega_f \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -10 & -3 \end{pmatrix}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix} \equiv \mathcal{A}\begin{pmatrix} v_i \\ R\omega_i \end{pmatrix}. \tag{19}$$

Note that

$$\mathcal{A}^2 = \frac{1}{49}\begin{pmatrix} 49 & 0 \\ 0 & 49 \end{pmatrix} = \mathcal{I}. \tag{20}$$

Let us now consider the effects of the tilted plane. Since the ball's speed perpendicular to the plane is unchanged by each bounce, the ball spends the same amount of time in the air between any two successive bounces. This time equals $T = 2V/g\cos\theta$, because the component of gravity perpendicular to the plane is $g\cos\theta$. During this time, the speed along the plane increases by $(g\sin\theta)T = 2V\tan\theta \equiv V_0$.

[1] We have divided out the trivial $\omega_f = \omega_i$ and $v_f = v_i$ solution, which corresponds to slipping motion on a frictionless plane. The nontrivial solution we will find shortly is the non-slipping one. Basically, to conserve energy, there must be no work done by friction. But since work is force times distance, this means that either the plane is frictionless, or that there is no relative motion between ball's contact point and the plane. Since we are given that the plane has friction, the latter (non-slipping) case must be the one we are concerned with.

Let $\mathbf{Q}$ denote the $(v, R\omega)$ vector at a given time (where $v$ denotes the velocity component parallel to the plane). The ball is initially projected with $\mathbf{Q} = \mathbf{0}$. Therefore, right before the first bounce, we have $\mathbf{Q}_1^{\mathbf{before}} = (V_0, 0) \equiv \mathbf{V}_0$. (We have used the fact that $\omega$ doesn't change while the ball is in the air.) Right after the first bounce, we have $\mathbf{Q}_1^{\mathbf{after}} = \mathcal{A}\mathbf{V}_0$. We then have $\mathbf{Q}_2^{\mathbf{before}} = \mathcal{A}\mathbf{V}_0 + \mathbf{V}_0$, and so $\mathbf{Q}_2^{\mathbf{after}} = \mathcal{A}(\mathcal{A}\mathbf{V}_0 + \mathbf{V}_0)$. Continuing in this manner, we see that

$$\begin{aligned} \mathbf{Q}_n^{\mathbf{before}} &= (\mathcal{A}^{n-1} + \cdots + \mathcal{A} + \mathcal{I})\mathbf{V}_0, \quad \text{and} \\ \mathbf{Q}_n^{\mathbf{after}} &= (\mathcal{A}^{n} + \cdots + \mathcal{A}^2 + \mathcal{A})\mathbf{V}_0. \end{aligned} \tag{21}$$

However, $\mathcal{A}^2 = \mathcal{I}$, so all the even powers of $\mathcal{A}$ equal $\mathcal{I}$. The value of $\mathbf{Q}$ after the $n$th bounce is therefore given by

$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \mathbf{Q}_n^{\mathbf{after}} &= \frac{n}{2}(\mathcal{A} + \mathcal{I})\mathbf{V}_0. \\ n \text{ odd} \quad \Longrightarrow \quad \mathbf{Q}_n^{\mathbf{after}} &= \frac{1}{2}\Big((n+1)\mathcal{A} + (n-1)\mathcal{I}\Big)\mathbf{V}_0. \end{aligned} \tag{22}$$

Using the value of $\mathcal{A}$ defined in eq. (19), we find

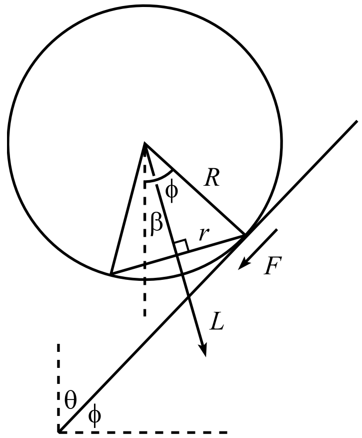
$$\begin{aligned} n \text{ even} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{n}{7}\begin{pmatrix} 5 & -2 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \\ n \text{ odd} \quad \Longrightarrow \quad \begin{pmatrix} v_n \\ R\omega_n \end{pmatrix} &= \frac{1}{7}\begin{pmatrix} 5n-2 & -2n-2 \\ -5n-5 & 2n-5 \end{pmatrix}\begin{pmatrix} V_0 \\ 0 \end{pmatrix}. \end{aligned} \tag{23}$$

Therefore, the speed along the plane after the $n$th bounce equals (using $V_0 \equiv 2V\tan\theta$)

$$\begin{aligned} v_n &= \frac{10nV\tan\theta}{7} \qquad (n \text{ even}), \\ v_n &= \frac{(10n-4)V\tan\theta}{7} \qquad (n \text{ odd}). \end{aligned} \tag{24}$$

Remark: Note that after an even number of bounces, eq. (23) gives $v = -R\omega$. This is the "rolling" condition. That is, the angular speed exactly matches up with the translation speed, so $v$ and $\omega$ are unaffected by the bounce. (The vector $(1, -1)$ is an eigenvector of $\mathcal{A}$.) At the instant that an even-$n$ bounce occurs, the $v$ and $\omega$ are the same as they would be for a ball that simply rolls down the plane. At the instant after an odd-$n$ bounce, the $v$ is smaller than it would be for a rolling ball, but the $\omega$ is larger. (And right before an odd-$n$ bounce, the $v$ is larger but the $\omega$ is smaller.)

6. It turns out that the ball can move arbitrarily fast around the cone. As we will see, the plane of the contact circle (represented by the chord in the figure below) will need to be tilted downward from the contact point, so that the angular momentum has a rightward horizontal component when it is at the position shown. In what follows, it will be convenient to work with the angle $\phi \equiv 90° - \theta$.




Let's first look at $F = ma$ along the plane. Let $\Omega$ be the angular frequency of the ball's motion around the cone. Then the ball's horizontal acceleration is $m\ell\Omega^2$ to the left. So $F = ma$ along the plane gives (where $F_f$ is the friction force)

$$mg\sin\phi + F_f = m\ell\Omega^2\cos\phi. \tag{25}$$

Now let's look at $\boldsymbol{\tau} = d\mathbf{L}/dt$. To get a handle on how fast the ball is spinning, consider what the setup looks like in the rotating frame in which the center of the ball is stationary (so the ball just spins in place as the cone spins around). Since there is no slipping, the contact points on the ball and the cone must have the same speed. That is,

$$\omega r = \Omega\ell \qquad \Longrightarrow \qquad \omega = \frac{\Omega\ell}{r}\,, \tag{26}$$

where $\omega$ is the angular speed of the ball in the rotating frame, and $r$ is the radius of the contact circle on the ball.[2] The angular momentum of the ball in the lab frame is $L = I\omega$ (at least for the purposes here[3]), and it points in the direction shown above.

The $\mathbf{L}$ vector precesses around a cone in $\mathbf{L}$-space with the same frequency, $\Omega$, as the ball moves around the cone. Only the horizontal component of $\mathbf{L}$ changes, and it traces out a circle of radius $L_{\text{hor}} = L\sin\beta$, at frequency $\Omega$. Therefore,

$$\left|\frac{d\mathbf{L}}{dt}\right| = L_{\text{hor}}\Omega = (I\omega\sin\beta)\Omega = \frac{I\Omega^2\ell\sin\beta}{r}\,, \tag{27}$$

and the direction of $d\mathbf{L}/dt$ is into the page.

The torque on the ball (relative to its center) is due to the friction force, $F_f$. Hence, $|\boldsymbol{\tau}| = F_f R$, and its direction is into the page. Therefore, $\boldsymbol{\tau} = d\mathbf{L}/dt$ gives (with

[2]If the center of the ball travels in a circle of radius $\ell$, then the $\ell$ here should actually be replaced with $\ell + R\sin\phi$, which is the radius of the contact circle on the cone. But since we're assuming that $R \ll \ell$, we can ignore the $R\sin\phi$ part.

[3]This $L = I\omega$ result isn't quite correct, because the angular velocity of the ball in the lab frame equals the angular velocity in the rotating frame (which tilts downwards with the magnitude $\omega$ we just found) plus the angular velocity of the rotating frame with respect to the lab frame (which points straight up with magnitude $\Omega$). This second part of the angular velocity simply yields an additional vertical component of the angular momentum. But the vertical component of $\mathbf{L}$ doesn't change with time as the ball moves around the cone. It is therefore irrelevant, since we will be concerned only with $d\mathbf{L}/dt$ in what follows.

$I = \eta m R^2$, where $\eta = 2/5$ in this problem)

$$\begin{aligned} F_f R &= \frac{I\Omega^2 \ell \sin\beta}{r} \\ \Longrightarrow \qquad F_f &= \frac{\eta m R\Omega^2 \ell \sin\beta}{r}. \end{aligned} \tag{28}$$

Using this $F_f$ in eq. (25) gives

$$mg\sin\phi + \frac{\eta m R\Omega^2 \ell \sin\beta}{r} = m\ell\Omega^2 \cos\phi. \tag{29}$$

Solving for $\Omega$ gives

$$\Omega^2 = \frac{g\sin\phi}{\ell\left(\cos\phi - \frac{\eta R\sin\beta}{r}\right)}. \tag{30}$$

We see that it is possible for the ball to move around the cone infinitely fast if

$$\cos\phi = \frac{\eta\sin\beta}{r/R}. \tag{31}$$

If we now define $\gamma \equiv \phi - \beta$ in the above figure, we have $r/R = \sin\gamma$, and $\beta = \phi - \gamma$. So eq. (31) becomes

$$\cos\phi = \frac{\eta\sin(\phi - \gamma)}{\sin\gamma}. \tag{32}$$

Using the sum formula for the sine in the numerator, we can rewrite this as

$$\begin{aligned} \tan\gamma = \frac{\eta}{1+\eta}\tan\phi &= \frac{2}{7}\tan\phi \\ &= \frac{2}{7}\cot\theta, \end{aligned} \tag{33}$$

where we have used $\eta = 2/5$. We finally obtain

$$\begin{aligned} \frac{r}{R} = \sin\gamma &= \frac{1}{\sqrt{1+\cot^2\gamma}} \\ &= \frac{1}{\sqrt{1+\frac{49}{4}\tan^2\theta}}. \end{aligned} \tag{34}$$

REMARKS:

(1) In the limit $\theta \approx 0$ (that is, a very thin cone), we obtain $r/R \approx 1$, which makes sense. The contact circle is essentially a horizontal great circle.

In the limit $\theta \approx 90^\circ$ (that is, a nearly flat plane), we obtain $r/R \approx 0$. The circle of contact points is very small, but the ball can still roll around the cone arbitrarily fast (assuming that there is sufficient friction). This isn't entirely intuitive.

(2) What value of $\phi$ allows the largest tilt angle of the contact circle (that is, the largest $\beta$)? From eq. (31), we see that maximizing $\beta$ is equivalent to maximizing $(r/R)\cos\phi$,

or equivalently $(r/R)^2 \cos^2\phi$. Using the value of $r/R$ from eq. (34), we see that we want to maximize

$$(r/R)^2 \cos^2\phi = \frac{\cos^2\phi}{1 + \frac{49}{4}\cot^2\phi}\,. \tag{35}$$

Taking the derivative with respect to $\phi$ and going through a bit of algebra, we find that the maximum is achieved when

$$\tan\phi = \sqrt{\frac{7}{2}} \qquad \Longrightarrow \qquad \phi = 61.9^\circ. \tag{36}$$

You can then show from eq. (31) that

$$\sin\beta_{\text{max}} = \frac{5}{9} \qquad \Longrightarrow \qquad \beta_{\text{max}} = 33.7^\circ. \tag{37}$$

(3) Let's consider three special cases for the contact circle, namely, a horizontal circle, a great circle, and a vertical circle.

(a) *Horizontal circle*: In this case, we have $\beta = 0$, so eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell}\,. \tag{38}$$

In this case, $\mathbf{L}$ points vertically, which means that $d\mathbf{L}/dt$ is zero, which means that the torque is zero, which means that the friction force is zero. Therefore, the ball moves around the cone with the same speed as a particle sliding without friction. (You can show that such a particle does indeed have $\Omega^2 = g\tan\phi/\ell$.) The horizontal contact-point circle ($\beta = 0$) is the cutoff case between the sphere moving faster or slower than a frictionless particle.

(b) *Great circle*: In this case, we have $r = R$ and $\beta = -(90^\circ - \phi)$. Hence, $\sin\beta = -\cos\phi$, and eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell(1+\eta)}\,. \tag{39}$$

This reduces to the frictionless-particle case when $\eta = 0$, as it should.

(c) *Vertical circle*: In this case, we have $r = R\cos\phi$ and $\beta = -90^\circ$, so eq. (30) gives

$$\Omega^2 = \frac{g\tan\phi}{\ell\left(1 + \frac{\eta}{\cos^2\phi}\right)}\,. \tag{40}$$

Again, this reduces to the frictionless-particle case when $\eta = 0$, as it should. But for $\phi \to 90^\circ$ (thin cone), $\Omega$ goes to zero, whereas in the other two cases above, $\Omega$ goes to $\infty$.